Orthometric Heights from GPS: The integrated Approach

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Abstract. A solution strategy for the determination of orthometric heights from GPS baselines, taking into account additional gravity field information as well as leveling data is presented. The strategy is based on integrated geodetic modeling, where the gravity field parameters are treated as signals. The emphasis is given on the creation of the signal covariance matrix. The method is illustrated using GPS/leveling data at Volvi (N–E Thessaloniki).

Keywords. GPS Observations, Integrated Geodesy, Orthometric Heights, Covariance functions.

1 Introduction

One of the most interesting task in the field of surveying is the accurate determination of the leveling heights from GPS observations. This work intend to the computation of orthometric heights using GPS Baselines, Gravity anomalies and height differences using integrated mathematical models. The strategy is based on integrated geodesy where the gravity field parameters (disturbing gravity values, geoidal heights or disturding potentials) are treated as signals.

Integrated geodesy has been introduced for the rigorous adjustment of observations with both geometric and gravimetric information using precise mathematical models. Furthermore, integrated geodesy is a method for the adjustment of observations depending not only in discrete parameters but also on unknown functions.

The basic techniques described here are based on the work of Hein (1981a,b) and Rossikopoulos (1986). A review of work in classical integrated geodesy, following the pioneer ideas of T. Krarup can be found in Hein (1986). First applications of relevant concepts in the estimation of orthometric heigts from GPS observations have been presented by Hein (1985), Hein et all. (1988) and Hatjidakis and Petridou (1999).

2 Notations

The following notation is used:

$\mathbf{r}_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}$: the coordinate vector ,

$\mathbf{r}_i^o$: the approximated coordinates

$\mathbf{r}_{ij}^b = \begin{bmatrix} \Delta X_{ij}^b \\ \Delta Y_{ij}^b \\ \Delta Z_{ij}^b \end{bmatrix}$: GPS baseline

ϕ_i, λ_i : geodetic latitude and longitude

H_i, h_i : orthometric and normal height

$U_i = U(\mathbf{r}_i)$: normal potential

$g_i = g(\mathbf{r}_i)$: gravity value

$\gamma_i = \gamma(\mathbf{r}_i)$: normal gravity field vector

$T_i = T(\mathbf{r}_i)$: disturbing potential

$N_i = \dfrac{T_i}{\gamma_i}$: geoid height

KM : Newton's gravitational attraction constant

$h_i = H_i + N_i = H_i + \dfrac{T_i}{\gamma_i}$

2 The observation equations

2.1 GPS observations

After the linearization and the assumption that the coordinates ϕ, λ are exactly known, the observations equations for the GPS baseline have the form (Hatjidakis and Petridou, 1999)

$$\mathbf{b}_{ij} = \mathbf{A}_{ij}\,\mathbf{x}_{ij} + \mathbf{B}_{ij}\,\mathbf{y} + \mathbf{G}_{ij}\,\mathbf{s}_{ij} + \mathbf{v}_{ij} \tag{1}$$

where

$$\mathbf{b}_{ij} = \mathbf{r}_{ij}^{b} - (\mathbf{r}_j^{o} - \mathbf{r}_i^{o}) - N_i^{o}\,\mathbf{m}_i + N_j^{o}\,\mathbf{m}_j \tag{2}$$

$\mathbf{A}_{ij}$, $\mathbf{x}_{ij}$ is the deterministic part

$$\mathbf{A}_{ij} = [-\,\mathbf{m}_i \quad \mathbf{m}_j\,] \;,\; \mathbf{x}_{ij} = \begin{bmatrix} \delta H_i \\ \delta H_j \end{bmatrix} \tag{3}$$

$\mathbf{y}$ is the vector of system transformation parameters (nuisance parameters)

$$\mathbf{y} = [\,\lambda \;\; \varepsilon_x \;\; \varepsilon_y \;\; \varepsilon_z\,]^T \tag{4}$$

$$\mathbf{B}_{ij} = [\,\mathbf{r}_{ij}^{o} \quad \mathbf{P}_1\,\mathbf{r}_{ij}^{o} \quad \mathbf{P}_2\,\mathbf{r}_{ij}^{o} \quad \mathbf{P}_3\,\mathbf{r}_{ij}^{o}] \tag{5}$$

$\mathbf{G}_{ij}$, $\mathbf{s}_{ij}$ is the stochastic part

$$\mathbf{G}_{ij} = [\,-\frac{1}{\gamma_i^{o}}\,\mathbf{m}_i \quad \frac{1}{\gamma_i^{o}}\,\mathbf{m}_j\,] \;,\; \mathbf{s}_{ij} = \begin{bmatrix} T_i \\ T_j \end{bmatrix} \tag{6}$$

and

$$\mathbf{m}_i = \begin{bmatrix} \cos\phi_i \cos\lambda_i \\ \cos\phi_i \sin\lambda_i \\ \sin\phi_i \end{bmatrix} \;,\; \mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{7}$$

$$\mathbf{P}_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{P}_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

2.2 Leveling observations (orthometric height differences)

For the leveling observations we have the simple equation

$$b_{ij} = \delta H_j - \delta H_i + v_{ij} \tag{9}$$

$$\text{where } b_{ij} = \Delta H_{ij}^{b} - \Delta H_{ij}^{o} \;. \tag{10}$$

2.3 Gravity observations

The observation equation for the magnitude of the earth's gravity field (gravity values) has the general form

$$g_i = \gamma_i^{o} - \mathbf{m}_i^T\,\mathbf{M}\,(\mathbf{r}_i - \mathbf{r}_i^{o}) - \mathbf{m}_i^T\,\mathrm{grad}T_i \tag{11}$$

The Marussi matrix $\mathbf{M}$ is given analytically in Rossikopoulos (1986) for various selections for a normal gravity field. For surveying applications we can use the simple model

$$U(\mathbf{r}_i) = \frac{KM}{R+z} \quad , \quad \gamma(\mathbf{r}_i) = \frac{KM}{(R+z)^2} \tag{12}$$

and

$$\mathbf{M} = \frac{\partial\gamma}{\partial\mathbf{r}_i}\bigg|_{o} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{2KM}{(R+z)^3} \end{bmatrix} \tag{13}$$

The observation equation for the above selection of normal gravity field takes the form

$$b_i = \left[-\frac{2KM}{(R+z_i)^3}\sin^2\phi_i \right]_o [\delta H_i] + \left[1 \quad -\frac{1}{\gamma_i}\frac{2KM}{(R+z_i)^3}\sin^2\phi_i \right]_o \begin{bmatrix} \delta g_i \\ T_i \end{bmatrix} + v_i \tag{14}$$

$$\text{where } b_i = g_i^{b} - \gamma_i - N_i^{o}\,\frac{2KM}{(R+z_i)^3}\sin^2\phi_i \tag{15}$$

2.4 Potential differences

Observed height differences ΔH are converted to potential differences $\Delta W = -g\,\Delta H$, utilizing observed or independently gravity values. The observation equation has the form

$$b_{ij} = \begin{bmatrix} -\gamma_i^{o} & \gamma_j^{o} \end{bmatrix} \begin{bmatrix} \delta H_i \\ \delta H_j \end{bmatrix} + T_j - T_i + v_{ij} \tag{16}$$

$$\text{where } b_{ij} = \Delta W_{ij} - U(\mathbf{r}_j^{o}) + U(\mathbf{r}_i^{o})\,. \tag{17}$$

Table 1: Local covariance models

	For gravity anomaly $K_{\Delta g}(S)$	For disturbing potential $K(S,z)$
Exponential model	$\sigma_g^2 e^{-\frac{S^2}{2d^2}}$	$C_E(-1,0)$
Reilly model	$\sigma_g^2 (1-\frac{S^2}{2d^2}) e^{-\frac{S^2}{2d^2}}$	$C_R(1,0)$
Moritz model	$\sigma_g^2 d^3 \frac{2d^2 - S^2}{2\sqrt{(S^2+d^2)^5}}$	$\sigma_g^2 \frac{d^3}{2\sqrt{S^2+(z+d)^2}}$
Poisson model	$\sigma_g^2 d^5 \frac{6d^2 - 9S^2}{2\sqrt{(S^2+d^2)^7}}$	$\sigma_g^2 \frac{(z+d)d^4}{6\sqrt{(S^2+(z+d)^2)^3}}$

$$C_E(q,m) = \sigma_g^2 (\frac{d}{\sqrt{2}})^{1-q} \frac{\rho^m}{m!} \sum_{k=0}^{1} \left\{ \Gamma(\frac{m+q+k+1}{2}) \, {}_1F_1(\frac{m+q+k+1}{2}; m+1; -\rho^2) \frac{(-\zeta)^k}{k!} \right\}$$

$$C_R(q,m) = \frac{d^2}{2} C_E(q,m), \quad \rho = \frac{S}{\sqrt{2}d}, \quad \zeta = \frac{(z_i + z_j)\sqrt{2}}{d}$$

$${}_1F_1(a;c;x) = 1 + \frac{a}{c}x + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \ldots \qquad \text{(Abramowitz and Stegun, 1970)}$$

$$\Gamma(x) = (x-1)! \quad \text{όταν} \;\; x \succ 1 \;\; \forall x \in Z \quad \text{or} \quad \Gamma(x+\frac{1}{2}) = \frac{1\;3\;5\;\ldots\;(2\;x-1)}{2^x}\Gamma(\frac{1}{2})$$

3 The adjustment with the integrated model

All observations, including observations of the gravity field, can be analyzed simultaneously with an integrated model

$$\mathbf{w} = \mathbf{A}\,\mathbf{x} + [\;\mathbf{G}\;\;\mathbf{I}\;] \begin{bmatrix} \mathbf{s} \\ \mathbf{v} \end{bmatrix} \qquad (18)$$

where **x** contains the deterministic parameters, **s** contains all the stochastic parameters and **v** are the observational errors. The adjustment problem is one of estimation with respect to **x** and prediction with respect to **s** and **v**. For the stochastic parameters it is assumed that their means

$$E\{\mathbf{s}\} = \boldsymbol{\mu},\; E\{\mathbf{v}\} = \mathbf{0} \qquad (19)$$

and the covariance matrices

$$E\{\mathbf{v}\mathbf{v}^T\} = \mathbf{C} \qquad (20)$$

$$E\{(\mathbf{s}-\boldsymbol{\mu})\,(\mathbf{s}-\boldsymbol{\mu})^T\} = \mathbf{K} \qquad (21)$$

$$\text{and } E\{(\mathbf{s}-\boldsymbol{\mu})\,\mathbf{v}^T\} = \mathbf{0}. \qquad (22)$$

The adjustment of the observations is carried out by applying the least squares principle

$$\mathbf{v}^T \mathbf{C}^{-1} \mathbf{v} + \mathbf{s}^T \mathbf{K}^{-1} \mathbf{s} = \min. \qquad (23)$$

which leads to best linear unbiased estimates for the determinist parameters **x** and best linear unbiased predictions for the stochastic ones **s**, **v**. The system of normal equations has the form

Table 2: The partial derivatives of signal covariance funcion

Exponential	$K' = -S\,C_E(1{,}0) - \dfrac{S^3\left[C_E(1{,}0) + C_E(1{,}2)\cos 2\alpha\right]}{2\left[(y_j - y_i)^2 - (x_j - x_i)^2\right]} + \dfrac{S\,C_E(1{,}0)\,(y_j - y_i)^2}{(y_j - y_i)^2 - (x_j - x_i)^2}$ $K'' = \dfrac{S^2\left[C_E(1{,}0) + C_E(1{,}2)\cos 2\alpha\right]}{2\left[(y_j - y_i)^2 - (x_j - x_i)^2\right]} + \dfrac{C_E(1{,}0)\,(y_j - y_i)^2}{(y_j - y_i)^2 - (x_j - x_i)^2}$, $\dfrac{\partial K}{\partial z_i} = C_E(0{,}0)$
Reily	$C_R(q+2, m) = C_E(q, m)$.
Moritz	$K' = -\dfrac{\sigma_g^2 d^3 S}{2(S^2 + (d+z)^2)^{1.5}}$, $K'' = \dfrac{\sigma_g^2 d^3\left[2S^2 - (d+z)^2\right]}{2(S^2 + (d+z)^2)^{2.5}}$, $\dfrac{\partial K}{\partial z_i} = -\dfrac{\sigma_g^2 d^3 (d+z)}{2(S^2 + (d+z)^2)^{1.5}}$
Poisson	$K' = -\dfrac{\sigma_g^2 d^4 (d+z) S}{2(S^2 + (d+z)^2)^{2.5}}$, $K'' = \dfrac{\sigma_g^2 d^4 (d+z)\left[2S^2 - 0.5(d+z)^2\right]}{(S^2 + (d+z)^2)^{3.5}}$ $\dfrac{\partial K}{\partial z_i} = \dfrac{\sigma_g^2 d^4\left[0.5S^2 - (d+z)^2\right]}{3(S^2 + (d+z)^2)^{2.5}}$

$$\begin{bmatrix} \mathbf{N_x} & \mathbf{N_{xs}} \\ \mathbf{N_{xs}^T} & \mathbf{N_s} + \mathbf{K}^{-1} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} \mathbf{u_x} \\ \mathbf{u_s} \end{bmatrix} \tag{24}$$

where

$$\mathbf{N_x} = \mathbf{A}^T \mathbf{C}^{-1} \mathbf{A}\,,\ \mathbf{N_s} = \mathbf{G}^T \mathbf{C}^{-1} \mathbf{G}\,,\ \mathbf{N_{xs}} = \mathbf{A}^T \mathbf{C}^{-1} \mathbf{G}$$

$$\text{and}\quad \mathbf{u_x} = \mathbf{A}^T \mathbf{C}^{-1} \mathbf{b}\,,\ \mathbf{u_s} = \mathbf{G}^T \mathbf{C}^{-1} \mathbf{b}\,. \tag{25}$$

The solution is given by the equations

$$\hat{\mathbf{x}} = \mathbf{R_x^g}\,[\,\mathbf{u_x} - \mathbf{N_{xs}} \mathbf{N_s^{-1}}\,\mathbf{u_s}\,]$$

$$\hat{\mathbf{s}} = \mathbf{N_s^{-1}}\,[\,\mathbf{u_s} - \mathbf{N_{xs}^T}\,\hat{\mathbf{x}}] \tag{26}$$

$$\text{where}\quad \mathbf{R_x} = \mathbf{N_x} - \mathbf{N_{xs}} \mathbf{N_s^{-1}}\,\mathbf{N_{xs}^T} \tag{27}$$

$$\text{and}\quad \mathbf{Q_{\hat{x}}} = \mathbf{R_x^g}$$

$$\mathbf{Q_{\hat{s}}} = \mathbf{N_s^{-1}} + \mathbf{N_s^{-1}}\,\mathbf{N_{xs}^T}\,\mathbf{Q_{\hat{x}}}\,\mathbf{N_{xs}}\,\mathbf{N_s^{-1}}$$

$$\mathbf{Q_{\hat{x}\hat{s}}} = -\,\mathbf{Q_{\hat{x}}}\,\mathbf{N_{xs}}\,\mathbf{N_s^{-1}} \tag{28}$$

Estimates of the covariance matrices of estimate parameters $\hat{\mathbf{x}}$, $\hat{\mathbf{s}}$ can be obtain by multiplying the cofactor matrices $\mathbf{Q_{\hat{x}}}$, $\mathbf{Q_{\hat{s}}}$ and $\mathbf{Q_{\hat{x}\hat{s}}}$ with the estimated a-posteriori variance

$$\hat{\sigma}^2 = \frac{\hat{\mathbf{v}}^T \mathbf{P}\,\hat{\mathbf{v}} + \hat{\mathbf{s}}^T \mathbf{K}^{-1}\,\hat{\mathbf{s}}}{n - r} \tag{29}$$

where $f = n - r$ are the degrees of freedom, n is the number of observations and r the number of deterministic parameters **x**.

More details on algorithms which can be used for the adjustment of (18) can be found in Dermanis and Fotiou (1992).

4 The covariance functions of signals

The covariance matrix of signals **K** is obtain from covariance function K(P,Q) of disturbing potential at two different points P ans Q, by applying the law of covariance propagation to the functionals relating the signals with the disturbing potential.

Initially an empirical covariance function is determined from the gravity anomalies. Local covariance models for disturbing potential on the local plane extended to the subspace above the plane and the corresponding functions of the gravity anomaly are given in table 1. In this table z_i is the height

Table 3: The covariance functions of signals (for i = j)

	Exponential	Reilly	Moritz	Poisson
$\sigma(T_i, T_i)$	$C_E(-1,0)$	$C_R(1,0)$	$\frac{\sigma_g^2}{2}\frac{d}{d+z}d^2$	$\frac{\sigma_g^2}{6}(\frac{d}{d+z})^3 d$
$\sigma(\delta g_i, T_i)$	$-\frac{\partial K}{\partial z_i}$	$-\frac{\partial K}{\partial z_i}$	$\frac{\sigma_g^2}{2}(\frac{d}{d+z})^3(d+z)$	$\frac{\sigma_g^2}{3}(\frac{d}{d+z})^3 d$
$\sigma(\delta g_i, \delta g_i)$	$2\,C_E(1,0)$	$2C_R(3,0)$	$\sigma_g^2(\frac{d}{d+z})^3$	$\sigma_g^2(\frac{d}{d+z})^4$

$$C_E(q,m) = \sigma_g^2(\frac{d}{\sqrt{2}})^{1-q}\sum_{k=0}^{1}\left\{\Gamma(\frac{q+k+1}{2})\frac{(-\zeta)^k}{k!}\right\} \quad , \quad C_R(q,m) = \frac{d^2}{2}C_E(q,m)$$

from reference level of covariance function at point i and S is the horizontal distance between points i and j.

The covariances of signals are computed from the relations

$$\sigma(T_i, T_j) = K \tag{30}$$

$$\sigma(\delta g_i, T_j) = -\frac{\partial K}{\partial z_i} \tag{31}$$

$$\sigma(\delta g_i, \delta g_j) = -K'' - \frac{1}{S}K' \tag{32}$$

where z_i is i and S is the horizntal distance between points i and j. The partial derivatives K', K'' and $\frac{\partial K}{\partial z_i}$ are presented in table 2.

The problem it comes up is to find covariance functions for the same point ($\sigma(T_i, T_i)$, $\sigma(\delta g_i, T_i)$, $\sigma(\delta g_i, \delta g_i)$). The solution is presented in table 3.

Detailed discusion for the statistical behavior of the gravity field and various models of covariance functions are given in Meier (1981) and Rossikopoulos (1986).

5 Examples – Applications

Volvi network constists of 12 points located in N-E Thessaloniki. The GPS campaign was carried out in 1996 and 74 GPS baselines was computed in the area. A total of 576 gravity free air anomalies was available in a grid with a 0.5×0.5 resolution.

Table 4. Solution 1: Gps Baselines, Height Differences, Gravity Anomalies

	Approximated Orthometric Height H° (m)	Corrections δH (cm)	Adjusted Orthometric Height H (m)	Geoid Disturbances (m)
17	234.681	0.462	234.686	-166.686
3	381.979	-0.668	381.972	-136.696
5	42.630	-1.437	42.616	105.426
13	90.827	-0.290	90.824	2.686
15	83.429	-1.003	83.419	-146.491
8	255.022	-1.134	255.011	-103.219
7	90.351	-1.900	90.332	-186.929
16	81.216	0.121	81.217	-145.156
4	566.666	-1.642	566.650	-210.752
10	210.635		210.635	-60.914
9	254.337	-0.598	254.331	-45.952
12	90.320	-0.557	90.314	176.778

Table 5. Solution 2: Gps Baselines, Height Differences

	Approximated Orthometric Height H° (m)	Corrections δH (cm)	Adjusted Orthometric Height H (m)	Geoid Disturbances (m)
17	234.681	0.462	234.686	-165.771
3	381.979	-0.668	381.972	-135.923
5	42.630	-1.437	42.616	104.982
13	90.827	-0.290	90.824	2.761
15	83.429	-1.003	83.419	-145.667
8	255.022	-1.134	255.011	-102.616
7	90.351	-1.900	90.332	-185.906
16	81.216	0.121	81.217	-144.340
4	566.666	-1.642	566.650	-209.609
10	210.635		210.635	-60.527
9	254.337	-0.598	254.331	-45.634
12	90.320	-0.557	90.314	-175.805

Initially an empirical covariance function was determined from the gravity anomalies. The variance and the correlation length is presented below for the Moritz Model: Variance ($mgal^2$)=1256.80, Correltion length (m) = 12397.67

Three stochastic solutions are presented using Gps Baseline, Height Differences, Gravity Anomalies. Solution 1 is a combination of Gps Baselines, Height Differences, Gravity Anomalies. Solution 2 is an approach using only Gps Baselines and Height Differences. Solution 3 is a combination of Gps Baselines and Gravity Anomalies.

Table 6. Solution 3: Gps Baselines, Gravity Anomalies

	Approximated Orthometric Height H^o (m)	Corrections δH (cm)	Adjusted Orthometric Height H (m)	Geoid Disturbances (m)
17	234.681	-0.887	234.672	-0.303
3	381.979	-2.248	381.957	-0.282
5	42.630	-2.523	42.605	-0.360
13	90.827	-1.423	90.813	-0.352
15	83.429	-1.954	83.409	-0.313
8	255.022	-1.603	255.006	-0.350
7	90.351	-2.492	90.326	-0.328
16	81.216	-0.655	81.209	-0.333
4	566.666	-3.107	566.635	-0.288
10	210.635		210.635	-0.277
9	254.337	-0.928	254.328	-0.321
12	90.320	-1.451	90.305	-0.276

References

Dermanis, A. and A. Fotiou (1992): "Methods and applications of observation adjustment". Editions Ziti (in Greek).

Hatjidakis N. and E. Petridou (1999): "Vertical Networks with integrated geodetic models". Diploma thesis, Department of Geodesy and Surveying. AUTH, p. 84.

Hein, G. W. (1981a): "A Contribution to 3D-Operational Geodesy. Part 1: Principle and Observation Equations of Terrestrial Type". Int. Symp. On Geodetic Networks and Computations, DGK, Reihe B, Heft Nr. 258/VII (1982), pp. 31-64.

Hein, G. W. (1981a): "A Contribution to 3D-Operational Geodesy. Part 2: Concepts of solution". Int. Symp. On Geodetic Networks and Computations, DGK, Reihe B, Heft Nr. 258/VII (1982), pp. 65-85.

Hein, G. W. (1985): "Orthometric Height Determination using GPS observations and the Integrated geodesy adjustment model". NOAA Technical Report NOS 110 NGS 32, Rockville, MD

Hein, G. W., A. Leick and S. Lambert (1988): "Orthometric Height Determination using GPS and gravity field data" GPS'88 Conference on Engineering Applications of GPS Satellite Surveying Technology, May 11-14, 1988, Nashville, TN.

Meier, S. (1981): "Planar geodetic covariance functions". Reviews of Geophysics and Space Physics, vol. 19, No. 4, pp. 673-686.

Rossikopoulos, D. (1986): "Integrated Control Networks". Dissertation submitted for the Doctor of Engineering Degree, Department of Geodesy and Surveying. AUTH, p. 440 (in Greek).