Evaluation of Orthometric and Related Height Systems Using a Simulated Mountain Gravity Field

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Abstract. With the widespread computation of gravimetric geoid and quasi-geoid models for GPS height transformation, there is renewed interest in defining vertical datums using height systems that are fully compatible with GPS-derived heights. Although orthometric heights are commonly used for such purposes, computation of their true value from spirit leveling observations requires an exact knowledge of gravity along the (curved) plumbline above the geoid. Without direct gravity observations, several hypotheses must be made, and as such, numerous variants (approximations) of the true orthometric height have been derived.

This paper evaluates six height systems using a simulated (exact) gravity field from a well-defined prismoidal mountain mass for which the gravity field and plumblines are unambiguously defined. From this, the simulated (true) orthometric heights are computed and compared to various approximate orthometric heights. Geometric (ellipsoidal), normal, and dynamic heights are also evaluated. Comparisons of the height systems are made regarding their utility, accuracy, and compatibility with gravimetric geoid and quasi-geoid models.

Keywords. Vertical datum, height system, orthometric correction, geoid, gravity simulation

1 Introduction

In recent years, modernization of vertical datums and associated height systems has received considerable attention worldwide, particularly with regard to GPS-derived heights (e.g., Featherstone 2001, Hannah 2001, NGS 1998, Benciolini *et al.* 2001). Much of this renewed interest has been fuelled by the use of GPS for precise positioning, which in turn has led to increasingly refined geoid and quasi-geoid models for GPS height transformations.

Many height systems have been defined for different vertical datums throughout the world. Each system has advantages and disadvantages with regard to ease of computation, accuracy requirements, data availability, compatibility with GPS, needs of the user community, and, less obviously, the topographic setting in which the heights are used.

This paper evaluates six different height systems using a simulated gravity field consisting of a prismoidal mountain mass superimposed on an ambient gravity field. To gain insight into the purely theoretical performance of the height systems, they are evaluated as if all required data are perfectly known, i.e., in the absence of observational errors.

2 Heights

2.1 Geometric and gravimetric heights

The widespread use of GPS has made geometric (i.e., ellipsoidal) heights directly available to many users. Although not referenced to gravity, they represent a simple linear height above and perpendicular to an ellipsoidal reference surface.

Gravimetric heights are referenced to gravity and can be expressed in the familiar form $H = C / g$, where C is the geopotential number and g is some scalar value of gravity (cf. Heiskanen and Moritz 1967, chapter 4). The way in which g is defined determines the type of gravimetric height system. *Dynamic heights* are defined as $H_{\mathrm{D}} = C / g_{\mathrm{o}}$, where g_{o} is an arbitrary value of gravity (typically taken as normal gravity at 45° latitude). *Normal heights* are defined as $H_{\mathrm{N}} = C / \bar{\gamma}$, where $\bar{\gamma}$ is the mean value of normal gravity along the (curved) normal gravity plumbline,

$$\bar{\gamma} = \frac{1}{H_{\mathrm{N}}} \int_{0}^{H_{\mathrm{N}}} \gamma(z) \; dz \; . \tag{1}$$

Orthometric heights are defined as $H_{\mathrm{O}} = C / \bar{g}$, where $\bar{g}$ is the mean value of actual gravity along the Earth's (curved) plumbline between the geoid and topographic surface,

$$\bar{g} = \frac{1}{H_{\mathrm{O}}} \int_{0}^{H_{\mathrm{O}}} g(z) \; dz \; . \tag{2}$$

Since the true value of $\bar{g}$ cannot generally be determined, approximations are necessary, and several types of orthometric height have been proposed. The three types considered in this paper are described below.

Helmert orthometric heights are based on the following Poincaré-Prey relationship for mean gravity,

$$\bar{g}^{\mathrm{H}} = g + \tfrac{1}{2} F H_{\mathrm{O}} - 2\pi G\rho H_{\mathrm{O}}\,, \tag{3}$$

where g is the observed gravity at the topographic surface, F is the (linear) vertical 'free air' gradient of gravity, G is the universal gravitational constant, and ρ is the (constant) topographic density. The right-most term in Eq. (3) is the Bouguer plate gravity expression, which accounts for the topographic mass above the geoid but neglects terrain effects.

Mader orthometric heights are based on mean gravity (Mader 1954, Krakiwsky 1965) given by,

$$\bar{g}^{\mathrm{M}} = g + \tfrac{1}{2} F H_{\mathrm{O}} - 2\pi G\rho H_{\mathrm{O}} + \tfrac{1}{2}\left(\delta g^{\mathrm{T}} - \delta g_{\mathrm{o}}^{\mathrm{T}}\right), \tag{4}$$

where dg^{T} and $dg_{\mathrm{o}}^{\mathrm{T}}$ are terrain corrections applied at the topographic surface and the geoid, respectively. The implicit assumption is that gravity varies linearly above the geoid. For the simulated model in this paper, the gravitational effect of the terrain is known exactly, so there is no need to apply terrain corrections to a fictitious Bouguer plate. Instead, Eq. (4) is modified for a total terrain effect (Heiskanen and Moritz 1967, p.132),

$$\bar{g}^{\mathrm{M}} = g + \tfrac{1}{2} F H_{\mathrm{O}} - \tfrac{1}{2}\left(g^{\mathrm{T}} - g_{\mathrm{o}}^{\mathrm{T}}\right) \tag{5}$$

where g^{T} and $g_{\mathrm{o}}^{\mathrm{T}}$ are the vertical components of gravity due to the topographic mass at the ground surface and the geoid, respectively.

Niethammer orthometric heights are based on the following equation for mean gravity (Niethammer 1932, Rapp 1961, Krakiwsky 1965),

$$\bar{g}^{\mathrm{N}} = g + \tfrac{1}{2} F H_{\mathrm{O}} - 2\pi G\rho H_{\mathrm{O}} + \delta g^{\mathrm{T}} + \bar{\delta} g^{\mathrm{T}}\,, \tag{6}$$

where

$$\bar{\delta} g^{\mathrm{T}} = \frac{1}{H_{\mathrm{O}}} \int_{0}^{H_{\mathrm{O}}} \delta g^{\mathrm{T}}\, dH \tag{7}$$

is the integral mean terrain effect on the plumbline between the geoid and the topographic surface. As with Mader's method, Niethammer mean gravity is computed here using the total terrain effect,

$$\bar{g}^{\mathrm{N}} = g + \tfrac{1}{2} F H_{\mathrm{O}} - g^{\mathrm{T}} + \bar{g}^{\mathrm{T}}\,. \tag{8}$$

2.2 Leveled heights and their corrections

The leveled height difference between points A and B is computed in the model as

$$\Delta n_{\mathrm{AB}} = \frac{2\,\left(W_{\mathrm{B}} - W_{\mathrm{A}}\right)}{g_{\mathrm{A}} + g_{\mathrm{B}}}\,, \tag{9}$$

where W is the gravitational potential at the points. In order to account for path-dependence due to the non-parallelism of level surfaces, leveled height differences require corrections of the form (Heiskanen and Moritz 1967),

$$K_{\mathrm{AB}} = \sum_{\mathrm{A}}^{\mathrm{B}} \frac{g - \gamma_{\mathrm{o}}}{\gamma_{\mathrm{o}}} \Delta n + \frac{g_{\mathrm{A}} - \gamma_{\mathrm{o}}}{\gamma_{\mathrm{o}}} H_{\mathrm{A}} - \frac{g_{\mathrm{B}} - \gamma_{\mathrm{o}}}{\gamma_{\mathrm{o}}} H_{\mathrm{B}} \tag{10}$$

where the correction K_{AB} is added to the leveled height difference Δn_{AB}; g_{A} and g_{B} are taken from the denominator in the gravimetric height definition $H = C / g$; and H_{A} and H_{B} are the absolute heights in the system used.

3 Synthetic gravity model

A simplified 'flat Earth' gravity field model was used for this study because it allows computations of essentially unlimited spatial resolution, even to the sub-millimeter level. In contrast, a model derived from a spherical harmonic expansion of the Earth's gravity field would require degree and order of ~20 million to achieve a resolution of one meter.

The synthetic gravity model used is defined in a right-handed rectangular coordinate system, with the z-axis positive upward. In this 3D system, the total gravitational potential (W) is a superposition of the *normal* and *disturbing* potentials,

$$W(x,y,z) = U(x,y,z) + T(x,y,z)\,, \tag{11}$$

where U is the normal potential and T is the disturbing potential, due here solely to the superimposed topographic mass. The normal potential and its derivatives are defined as

$$U(x,y,z) = W_{\mathrm{o}} - \gamma_{\mathrm{o}} z + \tfrac{1}{2} F z^2 \tag{12a}$$

$$\frac{\partial U}{\partial z} = \gamma(z) = -\gamma_{\mathrm{o}} + Fz \qquad \left(\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0\right) \tag{12b}$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{\partial \gamma}{\partial z} = F \qquad \left(\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial y^2} = 0\right), \tag{12c}$$

and all higher derivatives are zero. The z-dimension of the synthetic model denotes the geometric height, and throughout this paper is considered equal to the ellipsoidal height, h. The following values were used for the planar normal gravity:

$W_0 \equiv U_0 \equiv 62{,}636{,}860\ \mathrm{m}^2\,\mathrm{s}^{-2}$ is the gravitational potential of the geoid (at $z = h = 0$);

$\gamma_0 \equiv 9.8\ \mathrm{m\,s}^{-2}$ is normal gravity at $z = h = 0$;

$F \equiv 3.086\times10^{-6}\ \mathrm{s}^{-2}$ is the vertical gravity gradient.

Although the normal potential field used in this model is simplistic, it is mathematically consistent and a reasonably good approximation of Earth's normal field for areas of limited spatial extent. The maximum error due to the flat Earth approximation is ±0.8 mGal/km horizontally, and the error due to the linear gravity gradient is about –0.6 mGal/km vertically (cf. Hackney and Featherstone, 2002).

The disturbing potential due to the gravitational effect of the topographic masses was modeled using rectangular prisms. In general, the gravitational potential of a prism of uniform density with its edges parallel to a 3D rectangular coordinate system is given by (Nagy *et al.*, 2000)

$$T(x,y,z) = G\rho\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}\frac{dx\ dy\ dz}{r}\ , \qquad (13)$$

where

$r = \sqrt{x^2 + y^2 + z^2}$

$G = 6.67259\times10^{-11}\ \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ (Moritz, 2000)

$\rho = 2670\ \mathrm{kg\,m}^{-3}$.

The results of integrating Eq. (13), and its derivatives, are given by Nagy *et al.* (2000) and will not be repeated here. However, two equations in that publication [Eqs. (11) and (12)] contain misprints, and so are given below as used in this model:

$$\frac{\partial T}{\partial x} = G\rho\left|\left|\left| y\ln(z+r) + z\ln(y+r) - x\tan^{-1}\frac{yz}{xr}\right|_{x_1}^{x_2}\right|_{y_1}^{y_2}\right|_{z_1}^{z_2}$$

and

$$\frac{\partial T}{\partial y} = G\rho\left|\left|\left| z\ln(x+r) + x\ln(z+r) - y\tan^{-1}\frac{xz}{yr}\right|_{x_1}^{x_2}\right|_{y_1}^{y_2}\right|_{z_1}^{z_2}$$

Only the potential and its first derivatives were used in the model. Since both are defined over the whole R^3, there were no mathematical existence problems due to discontinuous higher order derivatives.

The topographic model is intended to represent a 'mountain' on a scale consistent with the highest mountains on Earth. The geometry is that of an 8-km-high, stepped pyramid, where each step is a 2-km-wide, 500-m-high bench. It consists of 16 rectangular prisms, each 500 m high, and has a total mass of 2.91×10^{16} kg. The basal dimensions are $x = y = 62$ km, and its lower surface coincides with the xy-plane. The mountain is centered at $x = 0$, with sides parallel to the coordinate axes. 'Benchmarks' are located at $x = 0, 2, 4, \ldots, 50$ km. On the mountain, these occur at the midpoint of each bench on the pyramid. A cross-section (logarithmic scale) of the mountain is shown in Figure 1.

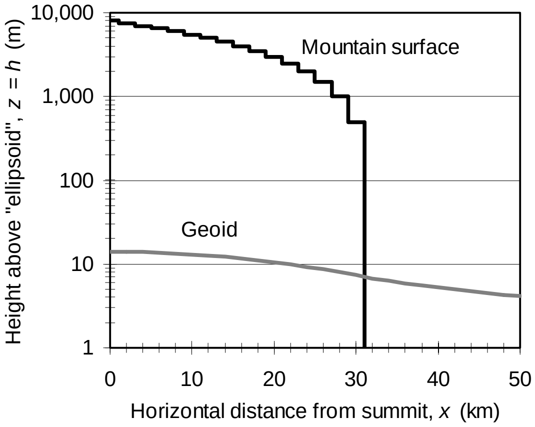

Fig. 1 Geometry of simulated mountain and resulting geoid.

3.1 Synthetic model computations

All computations were carried out in three dimensions, but the results used here were taken in a vertical plane of symmetry containing both the z and x axes. This simplified the display and analysis of results, for example by representing plumblines as plane curves rather than 3D curves.

The geoid height, N, resulting from superposition of the topographic mass on the planar normal field was computed by iteratively solving for the height $z = N$ while holding the potential fixed at the value of W_0. The resulting geoid surface is shown in Figure 1, where N ranges from 13.994 m at $x = 0$ to 4.108 m at $x = 50$ km. Helmert deflections of the vertical at the benchmarks (cf. Jekeli, 1999) are zero at $x = 0$, and increase to a maximum of 75.7" at $x = 22$ km. Although isostatic compensation of the simulated mountain is not modeled, the maximum deflection of the vertical in the model is consistent with observed values in excess of 70" near Mount Everest (Bomford 1971, p.528).

For computations performed 'at the geoid', the point on the geoid corresponding to the point on the topographic surface was taken along the straight ‘ellipsoid normal’, i.e., in a manner analogous to the Helmert projection. Comparisons with rigorously computed (Pizzetti) projections along the plumbline show that differences in heights are less than 1 mm, although differences in horizontal position are as large as 1.29 m.

Dynamic heights were computed in the model by taking $g_o = \gamma_0 = 9.8$ m s^{-2}. Normal heights were computed by integrating Eq. (1) with Eq. (12b) as the integrand and re-arranging to obtain

$$H_{\mathrm{N}} = \frac{\gamma_o}{F} - \sqrt{\left(\frac{\gamma_o}{F}\right)^2 - \frac{2C}{F}} \, . \tag{14}$$

‘True’ orthometric heights were computed using multiple line segments to approximate the (curved) plumbline between the synthetic geoid and the benchmarks. The segments were evaluated sequentially from the benchmark (where deflection of the vertical had been computed) down to the geoid in an iterative process such that each segment intersected perpendicular to an equipotential surface at its lower endpoint. The maximum line segment length was 130 m, and the error of ‘true’ orthometric heights is estimated as less than ±0.0001 mm.

For the computation of Niethammer orthometric heights, Eq. (7) was numerically integrated as

$$\bar{g}^{\mathrm{T}} = \frac{1}{H_{\mathrm{O}}} \sum_{i=1}^{k} \tfrac{1}{2}\left(g_{i-1}^{\mathrm{T}} + g_i^{\mathrm{T}} \right) L_i \, , \tag{15}$$

where g_i^{T} is the effect of gravity due to the terrain on the plumbline at the endpoint of a line segment, and k is the total number of segments of length L_i . The integration was performed along the ‘ellipsoid’ normal for all benchmarks, which is less computationally intensive than along the plumbline. As a check, integration was also performed along the plumbline for several benchmarks, and the resulting heights differed by less than ±0.1 mm.

4 Comparison of height systems

4.1 Orthometric height accuracy

Figure 2 shows the magnitude (i.e., absolute value) of the error in the various approximate orthometric heights (Section 2.2) with respect to the ‘true’ orthometric heights defined above. From Figure 2, Helmert heights show the largest errors, whereas errors in Niethammer heights are negligible. Mader height errors are of intermediate magnitude.

Both Helmert and Mader height errors show an apparent spike near the mountain edge. This is due to the errors going from positive to negative when plotted as absolute values. Although some numerical instability due to edge effects and the coarseness of the model may also contribute to these spikes, it should be noted that they are in the sub-millimeter range and are therefore of minor concern.

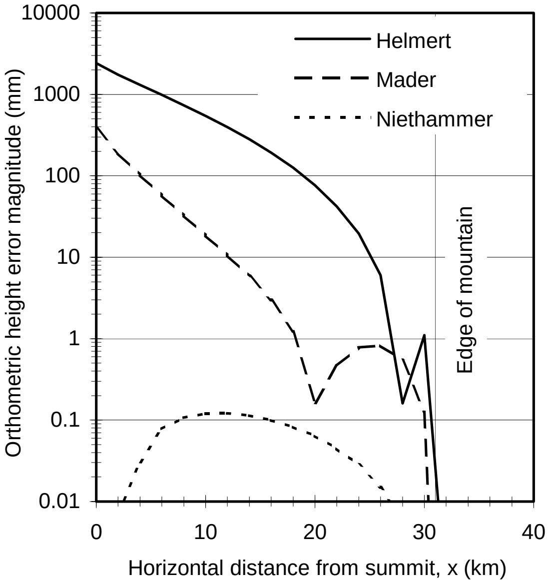

Fig. 2 Magnitude of error for various approximate orthometric heights with respect to ‘true’ orthometric heights

4.2 Relative height differences

Each of the height systems considered in this paper give absolute heights that can differ considerably from one another. However, the relative difference in heights is usually of greater interest than the absolute height. Relative height differences are illustrated in Figure 3 as the magnitude of the corrections to leveled heights between adjacent benchmarks, which were computed using Eq. (10) for gravimetric heights. For geometric heights, the correction was computed as the difference between the change in geometric height and the leveled height difference in Eq. (9). The difference in correction magnitudes between the various heights equals the difference in relative heights (except for dynamic heights, where the relative height difference equals the sum of the correction magnitudes, due to change in the sign of the corrections).

Figure 3 shows that corrections to leveled heights can be quite large. Geometric heights overall require the largest corrections. For gravimetric heights, the largest corrections apply to dynamic heights, and Helmert orthometric heights require corrections nearly as large. When the entire mountain is considered, Niethammer heights require the smallest corrections of all the height systems (although normal height corrections are slightly less for $x >$ ~18 km, $z <$ ~3500 m).

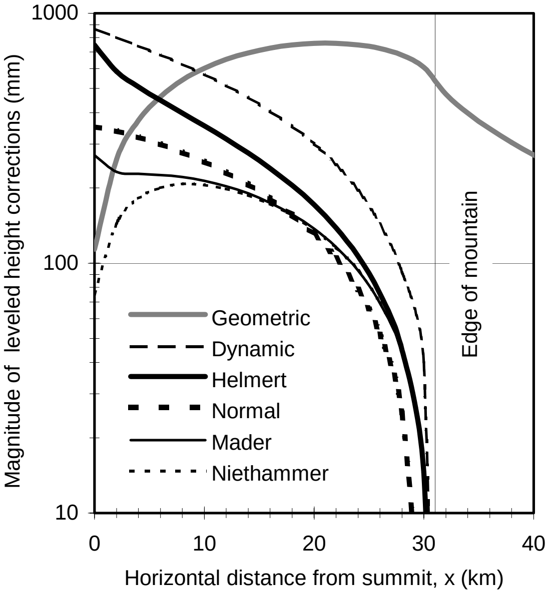

Fig. 3 Magnitude of corrections applied to leveled height differences between adjacent benchmarks.

5 Conclusions and discussion

Various height systems were evaluated using a synthetic model of the Earth's gravity field created by combining a planar normal field with that due to a pyramidal 'mountain' composed of right-rectangular prisms. Although simple, the model yielded results that could prove helpful in guiding selection of height systems to be used when 'modernizing' vertical datums. Summarized comparisons of the height systems evaluated in this study are given in Table 1.

The modeling results suggest that Niethammer orthometric and normal heights are theoretically consistent with geoid and quasi-geoid models, respectively, and are therefore considered appropriate for modern height systems. Mader orthometric heights are a more-or-less acceptable surrogate for Niethammer heights, but there seems little justification for their use given the continuing growth in computational power. Helmert orthometric heights are not recommended for modern height systems because of the large leveling corrections required and disagreement with true orthometric heights in mountainous terrain. In addition, since Helmert heights neglect gravimetric terrain corrections, they are theoretically incompatible with modern gravimetric geoid models, which may be part of the reason that 'hybrid' geoid models such as GEOID99 (Smith and Milbert, 2001) are necessary. Dynamic heights can always be made available as part of any height system that utilizes actual gravity data, but are generally not useful.

Additional research is recommended to aid in making informed decisions regarding development or modification of height systems, and to provide a theoretical foundation for their realization. Users of height data are understandably resistant to change, and therefore any changes must be justifiable and should occur infrequently. The choice to modify existing vertical datums cannot be made on theoretical grounds alone, but must also consider achievable accuracy, data availability, practicality, cost, and the overall needs of the user community.

Finally, the orthometric height systems considered here use a constant topographic mass density and employ terrain corrections that are not consistent with those used in geoid computation. As such, they are strictly not compatible with geoid models, particularly those computed using variable mass densities. Our future work will concentrate on the compatibility of different types of geoid models with different height systems.

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References

Benciolini B, Baldi P, Sacerdote F, Sansò F (2001) The height datum problem: the Italian case, *Int Geoid Service Bull*, 11: 118-136.

Bomford G (1971) *Geodesy*, 3rd edn, Oxford University Press, Great Britain, 731 pp.

Featherstone WE (2001) Prospects for the Australian Height Datum and geoid model, in: Adam J, Schwarz KP (Eds) *Vistas for Geodesy in the New Millennium*, Springer, Berlin, Germany, pp 96-101.

Hackney RI, Featherstone WE (2002) The gravity anomaly: geodetic and geophysical perspectives, *Geophys J Int* (submitted)

Hannah J (2001) An assessment of New Zealand's height systems and options for a future height datum, Report to Land Information New Zealand, 38 pp.

Heiskanen WA, Moritz H (1967) *Physical Geodesy*, WH Freeman & Co, San Francisco, USA, 364 pp.

Jekeli C (1999) An analysis of vertical deflections derived from high-degree spherical harmonic models, *J Geod*, 73: 10-22.

Krakiwsky EJ (1965) Heights, *M.S. Thesis*, Dept Geod Sci and Surv, Ohio State Univ, Columbus, USA, 157 pp.

Mader K (1954) The orthometric weight correction of precision leveling in high terrain, *Austrian Magazine for Surveying*, Special edn 15, Vienna, Austria (in German).

Moritz H (2000) Geodetic Reference System 1980, *J Geod* 74: 128-133.

Nagy D, Papp G, Benedek J (2000) The gravitational potential and its derivatives for the prism, *J Geod*, 74: 552-560.

NGS (1998) *National Height Modernization Study: Report to Congress*, National Geodetic Survey, Rockville, USA, 181 pp.

Niethammer T (1932) *Leveling and weight as a means for the computation of true sea level heights*, Swiss Geodetic Commission, Berne, Switzerland, 76 pp. (in German).

Rapp RH (1961) The orthometric height, *M.S. Thesis*, Dept Geod Sci, Ohio State Univ, Columbus, USA, 117 pp.

Smith DA, Roman DR (2001) GEOID99 and G99SSS: 1-arc-minute geoid models for the United States, *J Geod*, 75: 469-490.

Table 1. Comparison of height systems. Note that all gravimetric heights require observed gravity data, and all orthometric heights require an estimation of topographic mass density (which was assumed constant in this study).

Height system	**Advantages**	**Disadvantages**
Geometric (ellipsoidal)	▪ Clear geometric meaning ▪ Directly obtained from GPS measurements ▪ Gravity data not required ▪ Topographic density estimates and terrain corrections not required	▪ Not referenced to gravity, so of limited utility for applications involving fluid flow ▪ Requires overall largest corrections to leveling of all the height systems (cf. geoid or quasi-geoid heights) ▪ Not compatible with existing height infrastructure
Dynamic	▪ Simple to compute (no hypotheses about topographic density or terrain corrections required) ▪ Points with same height are on same level surface	▪ No geometric meaning, thus no simple way to derive from GPS measurements ▪ Requires largest corrections to leveling of all the gravimetric heights
Normal	▪ Simple to compute (no hypotheses about topographic density or terrain corrections required) ▪ Geometric meaning as height above quasi-geoid ▪ Quasi-geoid easier to compute than geoid ▪ Compatible with GPS when derived from quasi-geoid model ▪ Corrections to leveled heights usually small	▪ Quasi-geoid has no physical meaning (is not an equipotential surface) ▪ Corrections to leveling can become substantial in mountainous terrain
Helmert orthometric	▪ Simple to compute (no terrain corrections required) ▪ Can be derived from GPS using 'hybrid' geoid models ▪ Conceptually, terrain corrections are not needed in a geoid model used with this height system	▪ Poor agreement with true orthometric heights in mountainous terrain ▪ Large corrections to leveling often necessary ▪ Theoretically incompatible with gravimetric geoid (transformation of geoid may cause excessive falsification of geoid separation and slope)
Mader orthometric	▪ Good agreement with true orthometric heights in most situations ▪ Simpler to compute than Niethammer heights ▪ More consistent with gravimetric geoid than Helmert heights ▪ Corrections to leveled heights usually small	▪ Error increases in mountainous terrain ▪ Computationally intensive (requires computation of terrain effects at topographic surface and geoid) ▪ Assumes linear variation in gravity above geoid ▪ May require transformation of gravimetric geoid for consistency with GPS-derived heights
Niethammer orthometric	▪ Near perfect agreement with true orthometric heights in all situations ▪ More compatible with GPS-derived heights from a gravimetric geoid that includes terrain corrections ▪ Corrections to leveling are overall the smallest of all the height systems tested	▪ Most computationally intensive of all the height systems (but this should not be an issue given modern computer power)